Journal of Engineering Mathematics, Vol. 4, No. 4, October 1970
Wolters-Noordhoff Publishing–Groningen
Printed in the Netherlands

A Bi-Orthogonality Relation for Clamped Sector Plates

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(Received February 26, 1970 and in revised form June 19, 1970)

SUMMARY

Constructing a bi-orthogonality relation it is indicated how the solutions for the bending of a sector plate clamped along the radial edges and with physically meaningful boundary functions prescribed on the curved edge can be directly reduced to an infinite system of linear algebraic equations. Some numerical results are presented for the bending of a uniformly loaded clamped sector plate.

1. Introduction

The bending of a clamped sector plate under uniform load has been the subject of investigation by Carrier [1], Hasse [2], Carrier and Shaw [3], Conway and Huang [4], Morley [5] and a few others. Various approximate solutions to this problem have been suggested by these authors. Morley has given an exhaustive analysis of the existing literature on this problem and also its importance from the structural and mechanical engineering design points of view. Biharmonic eigen function expansions have been used by quite a few authors and mainly due to the non-availability of a suitable orthogonality relation it has not been possible to reduce the solution of the problem directly to a linear algebraic system of equations.

In the present paper we have constructed a bi-orthogonality relation for the eigen functions of the bi-harmonic equation in polar coordinates when the function and its derivative is made to vanish on the radial edges. Using this relation we have shown how the solutions for the bending of a sector plate clamped along the radial edges and with (a) deflection and slope or (b) deflection and moment or (c) moment and shear force prescribed on the curved edge can be reduced to an infinite system of linear algebraic equations. As an example we have considered the bending of a uniformly loaded clamped sector plate. We have presented some numerical results for the deflection of the central line when the semi-sector angle is 15°.

Similar bi-orthogonality relations for the eigen functions of the bi-harmonic equation for rectangular and cylindrical geometries can be found in [6], [7].

2. Basic Equations

We assume that the sector plate occupies the region $0 \leqq r \leqq 1$, $-\alpha \leqq \theta \leqq \alpha$ and that the radial edges $\theta = \pm\alpha$ are clamped. Let the functions $f^{(1)}(\theta)$, $f^{(2)}(\theta)$, $M_r(\theta)$ and $V_r(\theta)$ denote respectively the deflection, the slope, the moment and the shear force on the curved edge $r=1$. The governing differential equation and the boundary conditions are given by

$$\Delta\Delta w = 0 \qquad 0 \leqq r \leqq 1, \ -\alpha \leqq \theta \leqq \alpha \tag{2.1}$$

$$w = \frac{\partial w}{\partial \theta} = 0 \quad \text{on} \quad \theta = \pm\alpha . \tag{2.2}$$

On $r=1$ we have

$$w = f^{(1)}(\theta) \tag{2.3a}$$

$$\frac{\partial w}{\partial r} = f^{(2)}(\theta) \tag{2.3b}$$

$$M_r = -D\left[\frac{\partial^2 w}{\partial r^2} + \sigma\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right] \tag{2.3c}$$

and

$$V_r = -D\left[\frac{\partial}{\partial r}(\Delta w) + \frac{(1-\sigma)}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right)\right] \tag{2.3d}$$

where w denotes the deflection and Δ in polar coordinates (r, θ) is given by

$$\Delta \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

The constants σ and D denote the Poisson's ratio and the flexural rigidity respectively.

We assume that the solution of (2.1) is given by

$$w = \sum_n a_n r^{\lambda_n + 1} F_n(\theta). \tag{2.4}$$

Upon substituting in (2.1) and (2.2) we get

$$F_n^{iv} + 2(\lambda_n^2 + 1)F_n'' + (\lambda_n^2 - 1)^2 F_n = 0 \tag{2.5a}$$

with

$$F_n = F_n' = 0 \text{ on } \theta = \pm\alpha. \tag{2.5b}$$

The "even solutions" of (2.5) are given by

$$F_n(\theta) = \cos(\lambda_n + 1)\alpha \cos(\lambda_n - 1)\theta - \cos(\lambda_n - 1)\alpha \cos(\lambda_n + 1)\theta \tag{2.6}$$

where the λ_n's are the roots of the equation

$$\sin 2\lambda\alpha + \lambda \sin 2\alpha = 0. \tag{2.7}$$

It may be noted that the summation in (2.4) is extended over those eigenvalues of (2.7) whose real part is positive.

The roots of (2.7) which are symmetrically located over the four quadrants in the complex λ-plane can be calculated in any desired degree of accuracy using as the first approximation the asymptotic form for λ_n and then followed by Newton's iterative method.

The asymptotic form is given by

$$\lambda_n \simeq (4n-1)\frac{\pi}{4\alpha} + i\frac{1}{2\alpha}\log\left[(4n-1)\frac{\pi}{2\alpha}\sin 2\alpha\right].$$

The first ten roots of (2.7) have been calculated for $\alpha = 15°$.

3. Bi-Orthogonality Relation

We now define the operator Δ^* as follows:

$$\Delta^* \equiv \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Let us denote the values of $\Delta^* w$ and $\frac{\partial}{\partial r}(\Delta^* w)$ on $r=1$ by $f^{(3)}(\theta)$ and $f^{(4)}(\theta)$ respectively.

We now identify the functions $f^{(i)}(\theta)$, $(i=1, 2, 3, 4)$ related to F_n and F_n'' as follows

$$f^{(i)} = \Sigma a_n \Phi_n^{(i)}(\theta) \tag{3.1}$$

$$\Phi_n^{(1)} = F_n \tag{3.2a}$$

$$\Phi_n^{(2)} = (\lambda_n + 1)F_n \tag{3.2b}$$

$$\Phi_n^{(3)} = F_n'' + (\lambda_n^2 - 1)F_n \tag{3.2c}$$

$$\Phi_n^{(4)} = (\lambda_n - 1)F_n'' + (\lambda_n - 1)(\lambda_n^2 - 1)F_n. \tag{3.2d}$$

Let λ_m and λ_n be the eigenvalues corresponding to the eigenfunctions F_m and F_n respectively. Then we have from (2.5)

$$F_m^{iv}+2(\lambda_m^2+1)F_m''+(\lambda_m^2-1)^2F_m=0 \tag{3.3a}$$

$$F_n^{iv}+2(\lambda_n^2+1)F_n''+(\lambda_n^2-1)^2F_n=0 \tag{3.3b}$$

with

$$F_m=F_m'=F_n=F_n'=0 \text{ on } \theta=\pm\alpha\,. \tag{3.3c}$$

Upon multiplying (3.3a) by λ_n^2, (3.3b) by λ_m^2 subtracting and integrating by parts we get using (3.3c)

$$(\lambda_m^2-\lambda_n^2)\int_{-\alpha}^{+\alpha}[2F_m'F_n'-(\lambda_m^2+\lambda_n^2-2)F_mF_n]\,d\theta=0\,.$$

It follows that

$$\int_{-\alpha}^{+\alpha}[2F_m'F_n'-(\lambda_m^2+\lambda_n^2-2)F_mF_n]\,d\theta=0 \qquad (m\neq n) \tag{3.4}$$

We have

$$\int_{-\alpha}^{+\alpha}[\Phi_m^{(1)}\Phi_n^{(4)}+\Phi_m^{(2)}\Phi_n^{(3)}]\,d\theta=\int_{-\alpha}^{+\alpha}[F_m\{(\lambda_n-1)F_n''+(\lambda_n-1)(\lambda_n^2-1)F_n\}$$
$$+(\lambda_m+1)F_m\{F_n''+(\lambda_n^2-1)F_n\}]\,d\theta\,.$$

Integrating the right side by parts and upon using conditions (3.3c) it follows

$$\int_{-\alpha}^{+\alpha}[\Phi_m^{(1)}\Phi_n^{(4)}+\Phi_m^{(2)}\Phi_n^{(3)}]\,d\theta=-(\lambda_m+\lambda_n)\int_{-\alpha}^{+\alpha}[F_m'F_n'-(\lambda_n^2-1)F_mF_n]\,d\theta\,. \tag{3.5a}$$

Interchanging m and n in (3.5a) we get

$$\int_{-\alpha}^{+\alpha}[\Phi_m^{(3)}\Phi_n^{(2)}+\Phi_m^{(4)}\Phi_n^{(1)}]\,d\theta=-(\lambda_m+\lambda_n)\int_{-\alpha}^{+\alpha}[F_m'F_n'-(\lambda_m^2-1)F_mF_n]\,d\theta\,. \tag{3.5b}$$

Upon adding (3.5a) and (3.5b) we get

$$\int_{-\alpha}^{+\alpha}[\Phi_m^{(1)}\Phi_n^{(4)}+\Phi_m^{(2)}\Phi_n^{(3)}+\Phi_m^{(3)}\Phi_n^{(2)}+\Phi_m^{(4)}\Phi_n^{(1)}]\,d\theta$$
$$=-(\lambda_m+\lambda_n)\int_{-\alpha}^{+\alpha}[2F_m'F_n'-(\lambda_m^2+\lambda_n^2-2)F_mF_n]\,d\theta=0 \text{ for } m\neq n \text{ from equation (3.4)}\,.$$

It therefore follows

$$\int_{-\alpha}^{+\alpha}[\Phi_m^{(1)}\Phi_n^{(4)}+\Phi_m^{(2)}\Phi_n^{(3)}+\Phi_m^{(3)}\Phi_n^{(4)}+\Phi_m^{(4)}\Phi_n^{(1)}]\,d\theta=0 \qquad (m\neq n) \tag{3.6}$$

(3.6) is our required bi-orthogonality relation.

For $m=n$ it follows upon simplification that the left-hand side of (3.6) is

$$K_n=2\int_{-\alpha}^{+\alpha}[\Phi_n^{(1)}\Phi_n^{(4)}+\Phi_n^{(2)}\Phi_n^{(3)}]\,d\theta \tag{3.7a}$$

$$=-4\lambda_n(\cos 2\lambda_n\alpha+\cos 2\alpha)(2\alpha\cos 2\lambda_n\alpha+\sin 2\alpha)\,. \tag{3.7b}$$

4. Eigenfunction Expansions

Using (3.6) we can now find the constants a_n in the eigenfunction expansion (3.1) *viz.*,

$$f^{(i)}=\Sigma\, a_n\Phi_n^{(i)}(\theta) \qquad (i=1,2,3,4)\,.$$

It follows that

$$a_n K_n = \int_{-\alpha}^{+\alpha} [f^{(1)}\Phi_n^{(4)} + f^{(2)}\Phi_n^{(3)} + f^{(3)}\Phi_n^{(2)} + f^{(4)}\Phi_n^{(1)}]\, d\theta \,. \tag{4.1}$$

Using (2.3c) and (2.3d) it can easily be shown that $f^{(3)}(\theta)$ and $f^{(4)}\theta$ are related to $M_r(\theta)$ and $V_r(\theta)$ as follows.

$$f^3(\theta) = -\frac{1}{D} M_r - (1+\sigma) f^2(\theta) + (1-\sigma)\frac{d^2 f^{(1)}}{d\theta^2} \tag{4.2}$$

$$f^{(4)}(\theta) = -\frac{1}{D} V_r + \frac{2}{D} M_r + 2(1+\sigma) f^{(2)} - (1-\sigma)\frac{d^2 f^{(2)}}{d\theta^2} + (1+\sigma)\frac{d^2 f^{(1)}}{d\theta^2} \tag{4.3}$$

Case (a)

When deflection and slope are prescribed on the curved edge $r=1$, we take

$$f^{(3)}(\theta) = \Sigma\, a_m \Phi_m^{(3)}(\theta)\,, \quad f^{(4)}(\theta) = \Sigma\, a_m \Phi_m^{(4)}(\theta)$$

and it follows from (4.1)

$$a_n K_n = \int_{-\alpha}^{+\alpha} [f^{(1)}\Phi_n^{(4)} + f^{(2)}\Phi_n^{(3)}]\, d\theta + \sum_m a_m \int_{-\alpha}^{+\alpha} [\Phi_n^{(1)}\Phi_m^{(4)} + \Phi_n^{(2)}\Phi_m^{(3)}]\, d\theta \,.$$

Using (3.7a) the above equation can now be written in the form

$$a_n = G_n + \sum_m a_m S_{mn} \tag{4.4}$$

where

$$G_n = \frac{1}{K_n}\int_{-\alpha}^{+\alpha} [f^{(1)}\Phi_n^{(4)} + f^{(2)}\Phi_n^{(3)}]\, d\theta \tag{4.5}$$

and

$$S_{mn} = \frac{1}{K_n}\int_{-\alpha}^{+\alpha} [\Phi_n^{(1)}\Phi_m^{(4)} + \Phi_n^{(2)}\Phi_m^{(3)}]\, d\theta \,. \tag{4.6}$$

Using (3.7a) we get

$$S_{nn} = \tfrac{1}{2}$$

$$S_{mn} = \frac{2\lambda_m(\lambda_m+\lambda_n)\sin 2\alpha}{\{(\lambda_m^2+\lambda_n^2-4)^2 - 4\lambda_m^2\lambda_n^2\}}\left[\frac{(\lambda_n^2-1)(\cos 2\lambda_m\alpha + \cos 2\alpha) - (\lambda_m^2-1)(\cos 2\lambda_n\alpha + \cos 2\alpha)}{(\cos 2\lambda_n\alpha + \cos 2\alpha)(2\alpha\cos 2\lambda_n\alpha + \sin 2\alpha)}\right]$$

$$(m \neq n)\,.$$

Case (b)

When deflection and moment are prescribed on the curved edge, we take

$$f^{(2)}(\theta) = \Sigma\, a_m \Phi_m^{(2)}\,; \quad f^{(4)}(\theta)\, \Sigma\, a_m \Phi_m^{(4)}$$

and using (4.2) we have

$$f^{(3)}(\theta) = -\frac{M_r(\theta)}{D} + (1-\sigma)\frac{d^2 f^{(1)}}{d\theta^2} - (1+\sigma)\,\Sigma\, a_m \Phi_m^{(2)}\,.$$

It follows from (4.1)

$$a_n = \frac{1}{K_n}\int_{-\alpha}^{+\alpha}\left\{f^{(1)}\Phi_n^{(4)} - \frac{M_r}{D}\Phi_n^{(2)} + (1-\sigma)\frac{d^2 f^{(1)}}{d\theta^2}\Phi_n^{(2)}\right\} d\theta$$

$$+\frac{1}{K_n}\Sigma\, a_m \int_{-\alpha}^{+\alpha}\{\Phi_m^{(2)}\Phi_n^{(3)} + \Phi_m^{(4)}\Phi_n^{(1)} - (1+\sigma)\Phi_m^{(2)}\Phi_n^{(2)}\}\, d\theta \,.$$

The above equation is in the same form as (4.4).

Case (c)

When moment and shear force are prescribed on the curved edge, we take

$$f^{(1)} = \Sigma\, a_m \Phi_m^{(1)}\ ;\quad f^{(2)} = \Sigma\, a_m \Phi_m^{(2)}$$

and using (4.2) and (4.3) we have

$$f^{(3)} = -\frac{M_r}{D} - (1+\sigma)\,\Sigma\, a_m \Phi_m^{(2)} + (1-\sigma)\,\Sigma\, a_m \frac{d^2\Phi_m^{(1)}}{d\theta^2}$$

$$f^{(4)} = -\frac{V_r}{D} + \frac{2M_r}{D} + 2(1+\sigma)\Sigma\, a_m \Phi_m^{(2)} + (1+\sigma)\,\Sigma\, a_m \frac{d^2\Phi_m^{(1)}}{d\theta^2} - (1-\sigma)\,\Sigma\, a_m \frac{d^2\Phi_m^{(2)}}{d\theta^2}\,.$$

It follows from (4.1)

$$\begin{aligned} a_n = {} & \frac{1}{K_n}\int_{-\alpha}^{+\alpha} \frac{1}{D}\left\{-M_r \Phi_m^{(2)} + (2M_r - V_r)\Phi_n^{(1)}\right\} d\theta \\ & + \frac{1}{K_n}\Sigma\, a_m \int_{-\alpha}^{+\alpha} \Bigg[\Phi_m^{(1)}\Phi_n^{(4)} + \Phi_m^{(2)}\Phi_n^{(3)} \\ & + (1-\sigma)\left\{\frac{d^2\Phi_m^{(1)}}{d\theta^2}\Phi_n^{(2)} - \Phi_n^{(1)}\frac{d^2\Phi_m^{(2)}}{d\theta^2}\right\} \\ & + (1+\sigma)\left\{2\Phi_m^{(2)}\Phi_n^{(1)} - \Phi_m^{(2)}\Phi_n^{(2)} + \frac{d^2\Phi_m^{(1)}}{d\theta^2}\Phi_n^{(1)}\right\}\Bigg] d\theta\,. \end{aligned}$$

The above equation is in the same form as the equation (4.4).

5. Uniformly Loaded Clamped Sector Plate

We shall now consider the particular case of a uniformly loaded clamped sector plate. The governing differential equation is

$$\Delta\Delta w' = \frac{q_0}{D} \tag{5.1}$$

where w' denotes the deflection, q_0 the load and D the flexural rigidity.

The boundary conditions along the clamped edge $r=1$ are

$$w' = \frac{\partial w'}{\partial r} = 0\,. \tag{5.2}$$

We take the solution of (5.1) in the form

$$w' = w + w_1 \tag{5.3}$$

where w is the solution of the homogeneous equation given by (2.1) and w_1 is the particular solution of (5.1) corresponding to uniform load q_0. The solution for w_1 satisfying the condition of clamped radial edges is given by

$$w_1 = \frac{q_0}{D}\,\frac{r^4}{64(2+\cos 4\alpha)}\left[2+\cos 4\alpha - 4\cos 2\alpha\cos 2\theta + \cos 4\theta\right]. \tag{5.4}$$

Using (5.2) and (5.3) we get on $r=1$

$$w = f^{(1)}(\theta) = -w_1(1,\theta) \tag{5.5a}$$

and

$$\frac{\partial w}{\partial r} = f^{(2)}(\theta) = -4w_1(1,\theta)\,. \tag{5.5b}$$

Substituting the above values of $f^{(1)}(\theta)$ and $f^{(2)}(\theta)$ in (4.4) we get upon the evaluation of the integral

$$G_n = \frac{\lambda_n(\lambda_n+3)\sin 2\alpha}{2(2+\cos 4\alpha)K_n}\left[\frac{1}{(\lambda_n^2-9)} - \frac{(\lambda_n^2-1)\cos 4\alpha}{(\lambda_n^2-9)(\lambda_n^2-25)} - \frac{12\cos 2\alpha(\cos 2\lambda_n\alpha+\cos 2\alpha)}{(\lambda_n^2-1)(\lambda_n^2-25)}\right].$$

Equation (4.4) is a system of infinite equations in infinitely many unknowns. By truncation the value of a_n can be determined to any desired degree of accuracy.

6. Numerical Results

To determine the constants a_n, we have truncated the system (4.4) at $m=5$, 10. We give below the deflection of the central line $\theta=0$ for $\alpha=15°$. The results we have obtained agree very well with those given by Morley [5].

The first ten roots of the equation $\sin 2\lambda\alpha+\lambda\sin 2\alpha=0$ are also given here.

The calculation for the bending moments have been carried out along the radial edges and the central line $\theta=0$.

Roots of $\sin 2\lambda\alpha+\lambda\sin 2\alpha=0$ for $\alpha=15°$

n	Roots λ_n		
1	8.06296	$+i$	4.20286
2	20.46721	$+i$	5.83600
3	32.61272		6.69310
4	44.69125		7.28117
5	56.74125		7.72996
6	68.77619		8.09308
7	80.80215		8.39808
8	92.82226		8.66103
9	104.83836		8.89214
10	116.85157		9.09829

Deflection of the central line $\theta=0$ for $\alpha=15°$ and $m=5$

r	$(D/q)\,10^5\,w'(1,0)$
0	0.000
0.125	0.005
0.250	0.088
0.375	0.449
0.500	1.430
0.625	3.178
0.750	4.571
0.875	3.085(3.083)[3.080]
1.000	0.005(0.007)[0.002]

The values in parentheses are those obtained by Morley through a variational principle for $m=5$ and the values in brackets are those obtained by the authors for $m=10$.

Bending moments on the radial edges and the central line $\theta=0$ for $\alpha=15°$ and $m=5$

r	$(10^3/q_0)M_r(r,0)$	$(10^3/q_0)M_\theta(r,0)$	$(10^3/q_0)M_\theta(r,\alpha)$
0	0	0	0
0.125	0.017	0.183	−0.391
0.250	0.065	0.730	−1.562
0.375	0.134	1.658	−3.545
0.500	0.393	3.009	−6.419
0.625	1.456	4.507	−9.554
0.750	3.306(3.304)[3.305]	5.020(5.018)[5.018]	−10.565(−10.564)[−10.563]
0.875	2.602(2.600)[2.600]	2.887(2.879)[2.882]	−6.323(−6.320)[−6.325]
1.000	−9.237(−9.025)[−9.018]	−2.828(−2.708)[−2.705]	0.120(0.137)[0.019]

Concluding Remarks

It is shown how the bi-orthogonality relation presented here can be used to obtain solutions for the bending if a sector plate clamped along the radial edges and under arbitrary loadings on the curved edge. The numerical results presented here for the bending of a uniformly loaded clamped sector plate shows remarkable agreement with those obtained by Morley using a variational principle. It is interesting to note that though the boundary condition on the curved edge is approximately satisfied a good part of the deflected surface is indeed well represented around that edge.

The authors believe that the bi-orthogonality relation presented here makes the "sector-problem" simple and direct.

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Journal of Engineering Math., Vol. 4 (1970) 361–367